ON THE BEST UPPER BOUND FOR PERMUTATIONS AVOIDING A PATTERN OF A GIVEN LENGTH

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ABSTRACT. Numerical evidence suggests that certain permutation patterns of length k are easier to avoid than any other patterns of that same length. We prove that these patterns are avoided by no more than $(2.25k^2)^n$ permutations of length n. In light of this, we conjecture that no pattern of length k is avoided by more than that many permutations of length n.

1. INTRODUCTION

1.1. **Upper Bounds for Pattern Avoiding Permutations.** The theory of pattern avoiding permutations has seen tremendous progress during the last two decades. The key definition is the following. Let $k \leq n$, let $p = p_1p_2\cdots p_n$ be a permutation of length n, and let $q = q_1q_2\cdots q_k$ be a permutation of length k. We say that p avoids q if there are no k indices $i_1 < i_2 < \cdots < i_k$ so that for all a and b, the inequality $p_{i_a} < p_{i_b}$ holds if and only if the inequality $q_a < q_b$ holds. For instance, $p = 2537164$ avoids $q = 1234$ because p does not contain an increasing subsequence of length four. See [3] for an overview of the main results on pattern avoiding permutations.

Let $S_n(q)$ be the number of permutations of length n (or, in what follows, n-permutations) that avoid the pattern q. Since the spectacular result of Adam Marcus and Gábor Tardos [11], it is known that for every pattern q, there exists a constant c_q so that the inequality $S_n(q) \leq c_q^n$ holds for all n. As there are only $k!$ patterns of length q, it follows that for all positive integers k, there exists a constant c_k so that for all patterns q of length k, the inequality

$$S_n(q) \leq c_k^n \tag{1}$$

holds for all positive integers n.

However, the quest of finding the *best* constant c_k is in (1), is wide open. The result of Marcus and Tardos [11] has only shown that $c_k \leq 15^{2k^4\cdot\binom{k^2}{k}}$. Josef Cibulka [8] has improved this bound by showing that $c_k \leq 2^{O(k\log k)}$, but even this bound seems to be very far from reality, as we will explain.

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Richard Arratia [2] has conjectured that $c_k = (k-1)^2$ is sufficient for all k, but this conjecture was refuted by Albert and al [1], who proved that if n is large enough, then $S_n(1324) \geq 9.42^n$.

1.2. **Layered Patterns.** A *layered* pattern is a pattern consisting of decreasing subsequences (the layers) so that the entries decrease within the layers but increase among the layers, as in 3215476. Equivalently, a pattern is layered if and only if it avoids both 231 and 312. Layered permutations are important since numerical evidence (computed first by Julian West [13] and later replicated by many others) supports the following conjecture.

Conjecture 1.1. *Let q be a non-layered pattern of length k, and let Q be a layered pattern of length k. Then for all positive integers n, the inequality*

$$S_n(q) \leq S_n(Q) \tag{2}$$

holds.

If Conjecture 1.1 holds, then any upper bound that we can prove for all *layered* patterns of length k is also an upper bound for *all* patterns of length k. This has motivated several attempts to find a constant CL_k so that $S_n(Q) \leq CL_k^n$ for all layered patterns Q of length k. It follows from results in [5], [6] and [7] that $CL_k \leq O(2^k)$. A much stronger, recent result of Anders Claesson, Vit Jelinek and Einar Steingrímsson [9] shows that $CL_k \leq 4k^2$ holds.

Let M_{2m} denote the pattern $132\cdots(2m-1)(2m-2)2m$, and let M_{2m-1} denote the pattern obtained from M_{2m} by removing the first entry and then relabeling. That is, $M_{2m-1} = 21\cdots(2m-2)(2m-3)(2m-1)$. So for instance, $M_3 = 213$, and $M_4 = 1324$, while $M_5 = 21435$, and $M_6 = 132546$. A different version of Conjecture 1.1, also supported by numerical evidence, is the following.

Conjecture 1.2. *Let $m \geq 2$. Then*

(A) *for all positive integers n, and for all patterns q of length $2m-1$, the inequality*
$$S_n(q) \leq S_n(M_{2m-1})$$
holds, and

(B) *for all positive integers n, and for all patterns q of length $2m$, the inequality*
$$S_n(q) \leq S_n(M_{2m})$$
holds.

In this paper, we will prove that if $k = 2m-1$, then the inequality $S_n(M_{2m-1}) \leq (2.25k^2)^n$ holds, and if $k = 2m$, then the inequality $S_n(M_{2m}) \leq (2.25k^2)^n$ holds. This means that if Conjecture 1.2 holds, then $c_k \leq 2.25k^2$. (In fact, we prove a slightly stronger upper bound for c_k.) Note that $c_k \geq (k-1)^2$ has been known since Amitaj Regev's paper [12].

Our proof will be inductive, but even the initial case of our induction will depend on a result that has only been recently proved.

2. Composing and Decomposing Patterns

The following useful definitions describe two simple but crucial ways in which patterns can be composed.

Definition 2.1. *Let q be a pattern of length k and let t be a pattern of length m. Then $q \oplus t$ is the pattern of length $k+m$ defined by*

$$(q \oplus t)_i = \begin{cases} q_i \text{ if } i \leq k, \\ \\ t_{i-k} + k \text{ if } i > k. \end{cases}$$

In other words, $q \oplus t$ is the concatenation of q and t so that all entries of t are increased by the size of q.

Example 2.2. *If $q = 3142$ and $t = 132$, then $q \oplus t = 3142576$.*

Definition 2.3. *Let q be a pattern of length k and let t be a pattern of length m. Then $q \ominus t$ is the pattern of length $k+m$ defined by*

$$(q \ominus t)_i = \begin{cases} q_i + m \text{ if } i \leq k, \\ \\ t_{i-k} \text{ if } i > k. \end{cases}$$

In other words, $q \ominus t$ is the concatenation of q and t so that all entries of q are increased by the size of t.

Example 2.4. *If $q = 3142$ and $t = 132$, then $q \ominus t = 6475132$.*

The following strong theorem of Claesson, Jelinek and Steingrímsson describes an important way in which permutations avoiding a long given pattern of a specific kind can be decomposed into two permutations, each of which avoids a shorter pattern.

Theorem 2.5. [9] *Let σ, τ, and ρ be three permutations. Let p be a permutation that avoids $\sigma \oplus (\tau \ominus 1) \oplus \rho$. Then it is possible to color each entry of p red or blue so that the red entries of p form a $\sigma \oplus (\tau \ominus 1)$-avoiding permutation and the blue entries of p form a $(\tau \ominus 1) \oplus \rho$-avoiding permutation.*

Example 2.6. *Let σ, τ, and ρ each be the one-element pattern 1. Then Theorem 2.5 says that it is possible to color the entries of a 1324-avoiding permutation red or blue so that the red entries form a 132-avoiding permutation and the blue entries form a 213-avoiding permutation.*

Proof. (of Theorem 2.5) Color the entries of p one by one, going left to right, according to the following three rules.

1. If coloring p_i red creates a red copy of $\sigma \oplus (\tau \ominus 1)$, then color p_i blue.
2. If p_i is larger than a blue entry on its left, then color p_i blue.
3. Otherwise color p_i red.

It is then proved in [9] that this coloring has the required properties. □

Definition 2.7. *The coloring defined in the preceding proof is called the* canonical *coloring of p (with respect to σ, τ, and ρ).*

Example 2.8. *Let σ, τ, and ρ each be the one-element pattern as in Example 2.6, and let $p = 3612745$. Then p is a 1324-avoiding permutation. In the canonical coloring of p with respect to σ, τ, and ρ, the red entries are 3, 6, 1, 2, and 7, while the blue entries are 4 and 5. It is easy to verify that the string of red entries avoids 132, while the string of blue entries avoids 213.*

3. An Inductive Argument

In this section, we will present an inductive argument that shows how M_{2m-1}-avoiding and M_{2m}-avoiding permutations can be injectively mapped into pairs of certain words. The precise statement will be made in Lemma 3.9. Even the initial steps of this argument are not obvious. The argument for M_3-avoiding permutations (that is, 213-avoiding permutations), is not surprising. The argument for M_4-avoiding permutations has only been recently found [4]. These two arguments are given in Section 3.1, before the general result is announced in Section 3.2.

3.1. **The Initial Steps.** Let $p = p_1p_2 \cdots p_n$. We say that p_i is a *right-to-left maximum* in p if it is larger than all entries on its right, that is, if $p_i > p_j$ for all $j > i$. We always consider p_n a right-to-left maximum, since the condition is vacuously true for that entry.

Let $V_2(n)$ be the set of all words of length n over the alphabet $\{0, 1\}$.

Proposition 3.1. *Let $p = p_1p_2 \cdots p_n \in Av_n(213)$, let $v(p_i) = 0$ if p_i is not a right-to-left maxium, and let $v(p_i) = 1$ if p_i is a right-to-left maximum. Set*

$$v(p) = v(p_1)v(p_2) \cdots v(p_n)$$

and

$$v'(p) = v(1)v(2) \cdots v(n).$$

Then the map $f_3 : Av_n(213) \to V_2(n) \times V_2(n)$ defined by $f(p) = (v(p), v'(p))$ is injective.

Example 3.2. *If $p = 35412$, then $v(p) = 01101$, and so $v'(p) = 01011$. Therefore, $f(35412) = (01101, 01011)$.*

Proof. Let $(v, v') \in V_2(n) \times V_2(n)$, and let us assume that there is a permutation $p \in Av_n(213)$ so that $f_3(p) = (v, v')$. Then the positions of the 1s in v reveal the positions in which p must have right-to-left maxima, and the positions of 1s in v' reveal which entries of p are right-to-left maxima. It follows from the definition of right-to-left maxima that the right-to-left maxima of p must be in decreasing order from left to right.

Once the right-to-left maxima of p are in place, there is only one way to insert the remaining entries in the remaining slots. Indeed, going from right to left, in each step we must place the largest eligible remaining entry (that is, the largest one whose insertion does not change the set of right-to-left maxima). Indeed, if at some point during this procedure we placed an entry

x instead of the eligible entry $y > x$, then y, x, and the closest right-to-left minimum on the right of x would form a 213-pattern. $\square$

Note that by trivial arguments based on symmetry, analogous results can be proved for 312-avoiding, 231-avoiding, and 132-avoiding permutations. (We will actually use that last class.)

Also note that $v'(p)$ is nothing but the letters of the word $v(p)$ rearranged according to the *inverse* p^{-1} of p. That is, if $p^{-1} = p_{a_1}p_{a_2}\cdots p_{a_n}$, then $v'(p) = v(p^{-1}) = v(p_{a_1})v(p_{a_2})\cdots v(p_{a_n})$.

We say that a word w over a finite alphabet has an *XY-factor* if there is a letter X in w that is immediately followed by a letter Y. For instance, the word 00011120 has no 02-factors. Let $p = p_1p_2\cdots p_n$. We say that p_i is a *left-to-right minimum* if p if $p_i < p_j$ for all $j < i$. In other words, a left-to-right minimum is an entry that is less than everything on its left.

Let $W_4(n)$ be the set of all words of length n over the alphabet $\{1, 2, 3, 4\}$ that contain no 32-factors. The following is a recent result of the present author.

Lemma 3.3. [4] *Let $p = p_1p_2\cdots p_n \in Av_n(1324)$. Consider the canonical decomposition of p into a 132-avoiding permutation of red entries and a 213-avoiding permutation of blue entries as given in Theorem 2.5. Furthermore, define the word $w(p)$ as follows.*

- *If p_i is a red entry that is a left-to-right minimum in the string of red entries, let $w(p_i) = 1$,*
- *if p_i is a red entry that is not a left-to-right minimum in the string of red entries, let $w(p_i) = 2$,*
- *if p_i is a blue entry that is not a right-to-left maximum in the string of red entries, let $w(p_i) = 3$, and*
- *if p_i is a blue entry that is a right-to-left maximum in the string of blue entries, let $w(p_i) = 4$.*

Set

$$w(p) = w(p_1)w(p_2)\cdots w(p_n)$$

and

$$w'(p) = w(1)w(2)\cdots w(n).$$

Then the map $f_4 : Av_n(1324) \to W_4(n)\times W_4(n)$ defined by $f_4(p) = (w(p), w'(p))$ is injective.

Example 3.4. *If $p = 3612745$, then we get $w(p) = 1212234$, and $w'(p) = 1213424$, and we can easily see that neither $w(p)$ nor $w'(p)$ contains a 32-factor.*

Proof. (of Lemma 3.3) Let $f_4(p) = (w(p), w'(p))$, and let us assume that $w(p)$ contains a 32-factor, that is, there exists an index i so that $w(p_i) = 3$ and $w(p_{i+1}) = 2$. That means that in particular, p_i is blue and p_{i+1} is red, so by the second rule of canonical coloring (as given in Theorem 2.5), $p_i > p_{i+1}$. As p_{i+1} is not a left-to-right minimum, there is an entry p_j with

$j < i$ so that $p_j < p_{i+1}$. Similarly, as p_i is not a right-to-left maximum, there is an entry p_ℓ with $\ell > i+1$ so that $p_\ell > p_i$. However, that means that $p_j p_i p_{i+1} p_\ell$ is a 1324-pattern, which is a contradiction. An analogous argument (see [4]) shows that $w'(p)$ also avoids 1324. So f_4 indeed maps into $W_4(n) \times W_4(n)$.

In order to see that f_4 is injective, we proceed in a way that is similar to (but slightly more complex than) the way in which we proceeded in the proof of Proposition 3.1. Let $(w, w') \in W_4(n) \times W_4(n)$, and let us assume that there exists $p \in Av_n(1324)$ so that $f_4(p) = (w, w')$. Then the positions of 3s and 4s in w reveal *where* the blue entries of p are, and the positions of 3s and 4s in w' reveal *what* the blue entries of p are. By Proposition 3.1, this is sufficient information to recover the entire string of blue entries, since that string is a 213-avoiding permutation. A dual argument works for the string of red entries, since the red entries form a 132-avoiding permutation. □

3.2. **The Induction Step.** In this part of our proof, we will often obtain an encoding of a long permutation by partitioning it into two parts, encoding each part by disjoint alphabets, then combining the two images into one. The following definition makes this concept more precise. If s is a substring of the permutation p, let $|s|$ denote the length (number of entries) of s.

Definition 3.5. *Let $p = p_1 p_2 \cdots p_n$ be a permutation, and let p' and p'' be two substrings of p so that each entry of p belongs to exactly one of p' and p''.*

Let us assume that $a(p')$ and $a'(p')$ are words of length $|p'|$ over a finite alphabet A, and $b(p'')$ and $b'(p'')$ are words of length $|p''|$ over a finite alphabet B that is disjoint from A.

Then the merge *of $a(p')$ and $b(p'')$ is the word $w(p) = w_1 w_2 \cdots w_n$ of length n over the finite alphabet $A \cup B$ whose ith letter w_i is obtained as follows.*

(1) *If p_i is the rth letter p', then w_i is equal to the rth letter of $a(p')$, and*
(2) *if p_i is the rth letter of p'', then w_i is the rth letter of $b(p'')$.*

Furthermore, the merge of $a'(p')$ and $b'(p'')$ is the word $w'(p) = w'_1 w'_2 \cdots w'_n$ obtained as follows.

(1) *If the entry i of p is the tth smallest entry of p', then w'_i is equal to the tth letter of $a'(p')$, and,*
(2) *if the entry i of p is the tth smallest entry of p'', then w'_i is equal to the tth letter of $b'(p'')$.*

Example 3.6. *Let $p = 178942365$, let $p' = 12$, and let $p'' = 7894365$. Let $a(p') = 11$, and let $a(p'') = 11$. Furthermore, let $b(p'') = 2222233$, and let $b'(p'') = 2233222$.*

Then we have $w(p) = 122221233$ and $w'(p) = 112233222$.

The following definition extends the notion of merges from words to functions in a natural way.

Definition 3.7. *Let p, p' and p'' be as in Definition 3.5, and let $f(p') = (v_1, v_2)$, and $g(p'') = (w_1, w_2)$, where the v_i are words over the finite alphabet A, and the w_i are words over the finite alphabet B that is disjoint from A. Then we say that the function h is the merge of f and g if $h(p) = (z_1, z_2)$, where z_1 is the merge of v_1 and w_1, and z_2 is the merge of v_2 and w_2.*

Example 3.8. *Let $p = 687912435$, let $p' = 612$, and let $p'' = 879435$. Let $f(p') = (000, 000)$, and let $g(p'') = 112112$. If h is the merge of f and g, then $h(p) = (011200112, 001120112)$.*

Recall that M_{2m} denotes the pattern $132 \cdots (2m-1)(2m-2)2m$, and M_{2m-1} denotes the pattern obtained from M_{2m} by removing the first entry and then relabeling. That is, $M_{2m-1} = 2143 \cdots (2m-2)(2m-3)(2m-1)$. So $M_4 = 1324$, while $M_5 = 21435$, and $M_6 = 132546$.

In order to make the statement and proof of the following lemma easier to follow, we make the following general remark about the *indices* used in the lemma. The lemma will describe injections from certain sets of q-avoiding permutations into sets of pairs of certain words. These injections will be denoted by f_k, where k is the length of q. The co-domains of the injections f_k will be denoted by V_a or W_a, where a denotes the length of the words in the co-domain of the f_k.

Lemma 3.9. *Let $m \geq 2$. Let $Av_n(M_t)$ denote the set of all M_t-avoiding n-permutations.*

(a) *Let $V_{3m-4}(n)$ denote the set of all words of length n over the alphabet $\{0, 1, 2, \cdots, 3m-5\}$ that do not have any $(3i)(3i-1)$-factors for any $i \geq 1$.*
   Then there is an injection
$$f_{2m-1} : Av_n(M_{2m-1}) \rightarrow V_{3m-4}(n) \times V_{3m-4}(n).$$
(b) *Let $W_{3m-2}(n)$ denote the set of all words of length n over the alphabet $\{1, 2, \cdots, 3m-2\}$ that do not have any $(3i)(3i-1)$-factors for any $i \geq 1$.*
   Then there is an injection
$$f_{2m} : Av_n(M_{2m}) \rightarrow W_{3m-2}(n) \times W_{3m-2}(n).$$

Proof. We prove the statements by induction on m. For $m = 2$, the statements are true. Indeed, for $m = 2$, statement (a) is just the statement of Proposition 3.1, and statement (b) is just the statement of Lemma 3.3.

Now let us assume that the statements are true for m, and let us prove them for $m+1$.

(a) First, we prove statement (a). Let $p \in Av_n(M_{2m+1})$. Color all entries of p that are the leftmost entry of an M_{2m}-pattern in p green, and color all other entries of p yellow. Then, by definition, the string of all yellow entries of p forms an M_{2m}-avoiding permutation p''. By part (b) of our induction hypothesis, the map f_{2m} injectively maps this permutation p'' into a pair of words $(w_1, w_2) \in W_{3m-2}(|p''|) \times$

$W_{3m-2}(|p''|)$. In order to define the image $f_{3m-1}(p) \in V_{3m-1} \times V_{3m-1}$, simply mark all green entries of p by the letter 0. Let p' be the string of all green entries, and, to keep consistency with Definition 3.5, let $g(p') = (00 \cdots 0, 00 \cdots 0)$, where both strings of 0s are of length $|p'|$. Then we define $f_{2m+1}(p)$ as the merge of $g(p')$ and $f_{2m}(p')$ as defined in Definition 3.7.

Example 3.10. *For $p = 687912435$, the reader is invited to revisit Example 3.8. With our current terminology, p' is the string of green entries, p'' is the string of yellow entries, and $h = f_5$.*

It is clear that $f_{2m+1}(p)$ indeed does not have a $(3i)(3i-1)$-factor for any positive integer i, since positive integers correspond to yellow entries of p, and the string of yellow entries avoids all such factors by the induction hypothesis.

In order to show that the map f_{2m+1} is injective, let us assume that $(v_1, v_2) \in V_{3m-1}(n) \times V_{3m-1}(n)$ equals $f_{2m+1}(p)$ for some p. Then the positions of the yellow entries of p are easy to recover, since these are the positions in which v_1 has a positive value. Similarly, the values of the yellow entries can be recovered as the positions in which v_2 has a positive entry. Once the place and values of the yellow entries of p are found, the order in which these yellow entries is unique since the map f_{2m} that is applied to the string of yellow entries is injective. So the injective property of f_{2m} will be proved if we can show that there is only one way to place the green entries into the remaining slots.

Let us fill the remaining slots with the green entries going right to left. We claim that in each step, we must insert the largest remaining green entry that is eligible to go into the given position (that is, that will start an M_{2m}-pattern if inserted there). Indeed, let us assume that in a given position, we do not proceed as described. That is, both x and y are eligible to be inserted in a given position P, but we insert x, even if $x < y$. The fact that both x and y are eligible to be inserted in P means that they both will be the first entry of an M_{2m}-pattern if inserted in P. Let these patterns be xM and yM'. Then we will create an M_{2m+1} pattern, namely the pattern yxM' when we eventually insert y somewhere on the left of x.

(b) Now we prove statement (b). Let $p \in Av_n(M_{2m+2})$. Then Theorem 2.5 (with $\sigma = 1$, $\tau = 1$ and $\rho = M_{2m-1}$) shows that it is possible to color the entries of p red or blue so that the red entries form a 132-avoiding permutation and the blue entries form an M_{2m+1}-avoiding permutation. Let us consider the canonical coloring that achieves this and is given in the proof of Theorem [9].

Now we encode the string p' of all red entries of p in a manner that is analogous to what we saw in Proposition 3.1. We can do so, since the p' is a 132-avoiding permutation. To be more precise, mark

each entry of p' that is a left-to-right minimum in p' by the letter 1. Mark all remaining letters of p' by the letter 2. Define the words $a(p')$ and $a'(p')$ as in Proposition 3.1. That is, let $p' = p'_1 p'_2 \cdots p'_r$, and let $v(p'_i) = 1$ if p'_i is a left-to-right minimum in p', and let $v(p'_i) = 2$ otherwise. Then set $a(p') = v(p'_1)v(p'_2)\cdots v(p'_r)$, and set $a'(p') = v(p'_{j_1})v(p'_{j_2})\cdots v(p'_{j_r})$, where $p_{j_1} < p_{j_2} < \cdots < p_{j_r}$.

The string p'' of blue entries of p forms an M_{2m+1}-avoiding permutation, so as we have just seen in the proof of statement (a), the string p'' can be injectively mapped into a pair of words $(b(p''), b'(p'')) \in V_{3m-1} \times V_{3m-1}$ by the function f_{2m+1}. Shift these letters by three, that is, turn each letter i into a letter $i+3$ for all i in $b(p'')$ and $b'(p'')$. Finally, define $f_{2m+2}(p) = (w, w')$, where w is the merge of $a(p')$ and $b(p'')$, and w' is the merge of $a'(p')$ and $b'(p'')$.

It is clear that $f_{2m+2}(p) = (w, w')$ is a pair of words of length n over the alphabet $\{1, 2, \cdots, n\}$. It directly follows from the induction hypothesis that neither w nor w' can contain a $(3i)(3i-1)$-factor for $i > 1$. There remains to show that neither w nor w' can contain a 32-factor. In order to see this, let us assume that w contains a 32-factor in its jth and $(j+1)$st positions. The *type* of an entry of a permutation is just the letter it is mapped into by f_{2m+2}. That means that in particular, p_j is blue and p_{j+1} is red, so, by the second rule of canonical colorings (see the proof of Theorem 2.5), $p_j > p_{j+1}$, since a blue entry cannot be followed by a larger red entry. As p_{j+1} is of type 2, it is not a left-to-right minimum, so there exists an index $d < j$ so that $p_d < p_{j+1}$. As p_j is of type 3, it is of type 0 in p'', so there is an M_{2n}-pattern $p_j P$ in p'', and so, in p, whose first entry is p_j. However, that means that $p_d p_j p_{j+1} P$ is an M_{2m+2}-pattern in p, which is a contradiction. So w cannot contain a 32-factor, and in an analogous way, nor can w'.

Finally, we must show that f_{2m+2} is injective. By now, the method we used should not come as a surprise. Given a pair of words $(w, w') \in V_{3m+1}(n) \times V_{3m+1}(n)$, we can recover the set and positions of the red entries, and the set of positions of the blue entries of p, since the red entries are the ones that are of type 1 or 2. After this, it follows from Proposition 3.1 that we can recover the string of the red entries, and it follows from part (a) of this Lemma that we can recover the string of the blue entries.

□

4. Computing the Upper bounds

All there is left to do in order to find upper bounds on the numbers $S_n(M_{2m})$ and $S_n(M_{2m-1})$ is to find upper bounds on the sizes of the sets into which the relevant permutations can be injectively mapped. It would be straightforward to simply find an upper bound on the *exponential growth*

rate of these sequences, but we will carry out the slightly more cumbersome (but conceptually not difficult) task of finding upper bounds for the sequence in the sense we described in the introduction.

Proposition 4.1. *For all integers $m \geq 2$, we have*

$$|W_{3m-2}(n)| = C_1(m) \cdot \beta_1^n + C_2(m)\beta_2^n,$$

where $\beta_1 = \frac{3m-2+\sqrt{9m^2-16m+8}}{2}$ *and* $\beta_2 = \frac{3m-2-\sqrt{9m^2-16m+8}}{2}$, *while* $C_1 = \frac{\beta_1}{\beta_1-\beta_2}$ *and* $C_2 = \frac{\beta_2}{\beta_2-\beta_1}$.

Proof. Let $b_0 = 1$, and let $|b_n = W_{3m-2}(n)|$ for $n \geq 1$. It is then easy to see that $b_1 = 3m-2$, and

$$b_n = (3m-2)b_{n-1} - (m-1)b_{n-2} \tag{3}$$

for $n \geq 2$. Indeed, if we take an element of $V_{3m-2}(n-1)$, and append one of our $3m-2$ letters to its end, we will get an element of $W_{3m-2}(n)$, except in the $(m-1)b_{n-2}$ cases in which the last two letters of the new word form one of the forbidden factors.

Introducting the generating function $B(x) = \sum_{n\geq 1} b_n x^n$, we can turn formula (3) into a functional equation. Solving that equation, we get

$$B(x) = \frac{1}{1-(3m-2)x+(m-1)x^2}. \tag{4}$$

Finding the roots $r_1 = \frac{3m-2+\sqrt{9m^2-16m+8}}{2(m-1)}$ and $r_2 = \frac{3m-2-\sqrt{9m^2-16m+8}}{2(m-1)}$ of the denominator of $B(x)$, we see that $B(x)$ can be converted to the partial fraction form

$$B(x) = \frac{C_1(m)}{1-\frac{x}{\beta_1}} + \frac{C_2(m)}{1-\frac{x}{\beta_2}},$$

and our claim is now routine to prove. □

Corollary 4.2. *For all* even *positive integers k, the inequality*

$$S_n(M_k) \leq (2.25k^2)^n$$

holds.

Proof. Let $k = 2m$. Part (b) of Lemma 3.9 inductively constructs an injective map $f_{2m} : Av_n(M_{2m}) \to W_{3m-2}(n) \times W_{3m-2}(n)$. That map is not bijective. Indeed, it is obvious from the definition of f_{2m} that if $f_{2m}(p) = (w, w')$, then both w and w' must start with the letter 1.

Therefore, we know that

$$S_n(M_k) \leq |W_{3m-2}(n-1)|^2 = \left(C_1(m)\beta_1^{n-1} + C_2(m)\beta_2^{n-1}\right)^2. \tag{5}$$

It is routine to verify that for all integers $m > 1$, the inequality $\beta_2 < 1$ holds. As $\beta_1\beta_2 = m-1$, this means that $\beta_1 > m-1$, and so $C_1 = \frac{\beta_1}{\beta_1-\beta_2}$, this implies the inequality $C_1 < 2$ for $m \geq 3$. This same inequality can be verified for $m = 2$, since in that case, $\beta_1 = 2+\sqrt{3}$, and $\beta_2 = 2-\sqrt{3}$.

Furthermore, $C_2 = \frac{\beta_2}{\beta_2 - \beta_1} < 0$ since the denominator is negative. Hence,(5) implies

$$S_n(M_k) \leq (2\beta_1^{n-1})^2 \leq \beta^{2n}.$$

It is easy to prove from the definition of β that for all integers $m > 1$, the inequality $\beta < 3m - 2$ holds. Therefore,

$$S_n(M_k) \leq \beta^{2n} < (3m-2)^{2n} = (1.5k-2)^{2n} = (2.25k^2 - 3k + 4)^n.$$

□

Proposition 4.3. *For all integers $m \geq 2$, we have*

$$|V_{3m-2}(n)| = K_1 \cdot \alpha_1^n + K_2 \cdot \alpha_2^n$$

where $\alpha_1 = \frac{3m-4+\sqrt{9m^2-28+24}}{2}$, *and* $\alpha_2 = \frac{3m-4-\sqrt{9m^2-28+24}}{2}$, *while* $K_1 = \frac{\alpha_1}{\alpha_1 - \alpha_2}$ *and* $K_2 = \frac{\alpha_2}{\alpha_2 - \alpha_1}$.

Proof. Analogous to that of Proposition 4.1. □

Corollary 4.4. *For all* odd *positive integers k, the inequality*

$$S_n(M_k) \leq (2.25k^2)^n$$

holds.

Proof. The proof is analogous to that of Corollary 4.2. The only difference is that now we set $k = 2m - 1$, and then we use part (a) of Lemma 3.9. We get that

$$S_n(M_k) \leq |V_{3m-4}(n-1)|^2 \leq \alpha_1^{2n}.$$

So

$$\begin{aligned} S_n(M_k) &\leq \alpha_1^{2n} \\ &< (3m-4)^{2n} \\ &= (1.5k - 2.5)^{2n} \\ &= (2.25k^2 - 7.5k + 6.25)^n \\ &\leq (2.25k^2)^n. \end{aligned}$$

□

References

[1] M. H. Albert, M. Elder, A. Rechnitzer, P. Westcott, M. Zabrocki, On the Stanley-Wilf limit of 4231-avoiding permutations and a conjecture of Arratia, *Adv. in Appl. Math.* **36** (2006), no. 2, 96–105.

[2] R. Arratia, On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern, *Electronic J. Combin.*, **6** (1999), no. 1, N1.

[3] M. Bóna, *Combinatorics of Permutations*, second edition, CRC Press - Chapman Hall, 2012.

[4] M. Bóna, A new upper bound for 1324-avoiding permutations, *Preprint, arXiv:1207.2379, July 10, 2012, submitted to CUP.*

[5] M. Bóna, A simple proof for the exponential upper bound for some tenacious patterns. Adv. in Appl. Math. 33 (2004), no. 1, 192198.

[6] M. Bóna, The limit of a Stanley-Wilf sequence is not always rational, and layered patterns beat monotone patterns. *J. Combin. Theory Ser. A* **110** (2005), no. 2, 223235.

[7] M. Bóna, Records in Stanley-Wilf limits, *European J. Combin.*, **28** (2007), no. 1, 75-85.

[8] J. Cibulka, On constants in the Füredi-Hajnal and the Stanley-Wilf conjecture. Journal of Combinatorial Theory, Series A, 116(2):290-302,

[9] A. Claesson, V. Jelinek, E. Steingrímsson, Upper bounds for the Stanley-Wilf limit of 1324 and other layered patterns. *Preprint, arXiv:1111.5736v1, November 24, 2011, submitted to Elsevier.*

[10] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[11] A. Marcus, G. Tardos, Excluded Permutation Matrices and the Stanley-Wilf conjecture, *J. Combin. Theory Ser. A* **107** (2004), no. 1, 153–160.

[12] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, Advances in Mathematics **41** (1981), 115-136.

[13] J. West, Permutations with forbidden subsequences; and, Stack sortable permutations, PHD-thesis, Massachusetts Institute of Technology, 1990.

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